Solutions Exam
Mathematical Models for the spread of infectious diseases (WMMA061-05)
Thursday February 1 2024, 11.45-13.45

1. Consider a supercritical standard SIR epidemic in a homogeneously mixing population. Assume that initially there are n susceptible people, $m = \lfloor \log n \rfloor$ infectious people and no removed individuals. The per pair infectious contact rate is λ/n and the infectious periods are distributed as L, and assume $Var(L) < \infty$. Let $Z^{(n)}(\infty)$ be the number of ultimately removed people and $z^{(n)} = Z^{(n)}(\infty)/n$. Below you may use without proof that $z^{(n)}$ converges in probability to some real number $z > 0$ as $n \to \infty$.

a) Provide an identity determining z, and justify your answer (a correct heuristic justification is enough). (15pt)

b)Let $W : [-e^{-1}, \infty) \mapsto [-1, \infty)$ be the principal branch of the Lambert W function. That is,

$$W(y) := \{w \in [-1, \infty); we^w = y\},$$

where you may use without proof that for all $y \in [-e^{-1}, \infty)$, $W(y)$ indeed has exactly one element in $[-1, \infty)$.

Show that $z = 1 + \frac{W(-Re^{-R})}{R}$, where $R = \lambda\mathbb{E}[L]$. (5pt)

Solution

a) Let L_i be the infectious period of the i-th person to be infectious (for the initial m infected people, it does not matter how they are attributed infectious period $L_1, L_2, \cdots, L_m$).

Consider person x, who is uniformly chosen out of the n initially susceptible people. We want to compute $z^{(n)}$ the fraction of initially susceptible people that does not avoid infection. This fraction is equal to

$$1 - q^{(n)} = 1 - \mathbb{P}(x \text{ is ultimately susceptible}).$$

Now using that $z^{(n)}$ converges in probability to a constant we obtain

$$\begin{aligned} q^{(n)} = \prod_{k=1}^{Z^{(n)}} e^{-(\lambda/n)L_k} &= \exp\left[-\frac{\lambda}{n}\sum_{k=1}^{Z^{(n)}} L_k\right] \\ &= \exp\left[-\lambda\frac{Z^{(n)}}{n}\frac{1}{Z^{(n)}}\sum_{k=1}^{Z^{(n)}} L_k\right] = \exp\left[-\lambda z^{(n)}\frac{1}{Z^{(n)}}\sum_{k=1}^{Z^{(n)}} L_k\right]. \end{aligned}$$

Here the first equation follows from that x avoids infection if and only if all of the infected people have zero contacts with x during their infectious period. $Z^{(n)}$ is at least $m = \lfloor \log n \rfloor \to \infty$ and the infectious period of a person is independent of the final size. So,

$$\frac{1}{Z^{(n)}}\sum_{k=1}^{Z^{(n)}} L_k \overset{a.s.}{\to} \mathbb{E}[L]$$

by the law of large numbers. This leads by taking limits on both sides to $q^{(n)} \overset{\mathbb{P}}{\to} 1 - z$ and $q^{(n)} \overset{\mathbb{P}}{\to} e^{-\lambda z \mathbb{E}[L]}$ to that z should satisfy

$$1 - z = e^{-Rz},$$

where for notational convenience we have written $R := \lambda\mathbb{E}[L]$.

b) We have $1 - z = e^{-Rz}$, which implies after bringing all terms containing z to the left hand side, that

$$-(1-z)Re^{-(1-z)R} = -Re^{-R},$$

which implies that

$$-(1-z)R = W(-Re^{-R}),$$

which in turn implies

$$z = 1 + \frac{W(-Re^{-R})}{R}.$$

2. Consider a supercritical Markov SIR epidemic on a Configuration Model network of N vertices with expections of the degrees $\mathbb{E}[D] = \mu$ and finite second moment of the degrees $\mathbb{E}[D^2] = \mu_2$. Assume that the infectious contact rate per pair of neighbors is λ and the infectious periods are distributed as L, which is exponentially distributed with expectation $1/\gamma$.

a) Compute the basic reproduction number R_0^{CM} for this model. (10pt)

b) Compute the Malthusian parameter α^{CM} and express the basic reproduction number R_0^{CM} in terms of μ, μ_2, γ and α^{CM}. (10pt)

c) Consider a homogeneously mixing population of size N in which the contact rate per pair of individuals is λ/N and Infectious period distribution is L as above. Compute also for this model the Malthusian parameter α^{HM} and the basic reproduction number R_0^{HM}. Then write also for this model the basic reproduction number R_0^{HM} as a function of the Malthusian parameter α^{HM} and of γ.

Finally show that for given $\alpha = \alpha^{CM} = \alpha^{HM} > 0$ and γ, $R_0^{HM} \geq R_0^{CM}$. (15pt)

Solution

a) The expected number of susceptible neighbours of a just infected vertex in the early stages of an epidemic is

$$\mathbb{E}[\bar{D}-1] = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]} = \frac{\mu_2 - \mu}{\mu} = +\frac{\mu_2}{\mu} - 1.$$

For reasons of conveniance we still use $\mathbb{E}[\bar{D}-1]$ for this expression. The probability that on a given Susceptible-Inftectious edge, an infectious contact occurs before recovery of the infectious is

$$\int_0^\infty \gamma e^{-\gamma t}(1-e^{-\lambda t})dt = \frac{\lambda}{\lambda+\gamma} =: \psi.$$

So $R_0^{CM} = \mathbb{E}[\bar{D}-1]\psi$.

b) Use the Lotka-Euler equation $1 = \int_0^\infty e^{-\alpha^{CM}t}\mu^{CM}(dt)$, where

$$\mu^{CM}(dt) = \mathbb{E}[\bar{D}-1]\lambda e^{-\lambda t}\mathbb{P}(L>t)dt = \mathbb{E}[\bar{D}-1]\lambda e^{-\lambda t}e^{-\gamma t}dt.$$

So, the Lotka Euler equation gives

$$1 = \mathbb{E}[\bar{D}-1]\frac{\lambda}{\alpha^{CM}+\lambda+\gamma}.$$

So, $\alpha^{CM} = (\mathbb{E}[\bar{D}-1]-1)\lambda - \gamma$. That is, $\lambda = \frac{\alpha+\gamma}{\mathbb{E}[\bar{D}-1]-1}$ and $\psi = \frac{\alpha+\gamma}{\alpha+\gamma\mathbb{E}[\bar{D}-1]}$.

From part (a) we obtain

$$R_0^{CM} = \mathbb{E}[\bar{D}-1]\psi = \mathbb{E}[\bar{D}-1]\frac{\alpha+\gamma}{\alpha+\gamma\mathbb{E}[\bar{D}-1]} = \frac{\alpha+\gamma}{\big(\alpha/\mathbb{E}[\bar{D}-1]\big)+\gamma}.$$

c) We know that for the Markov SIR epidemic in a homogeneously mixing population $\mu^{(HM)}(dt) = \lambda\mathbb{P}(L>t)dt$, and the Lotka Euler equation becomes

$$1 = \int_0^\infty e^{-\alpha^{HM}t}\lambda\mathbb{P}(L>t)dt = \int_0^\infty e^{-\alpha^{HM}t}\lambda e^{-\gamma t}dt = \frac{\lambda}{\alpha^{HM}+\gamma}$$

and $\alpha^{HM} = \lambda - \gamma$, while $R_0^{HM} = \int_0^\infty \mu^{HM}(dt) = \lambda/\gamma$. So, $R_0^{HM} = 1 + \alpha^{HM}/\gamma$.

Now if $\alpha = \alpha^{CM} = \alpha^{HM}$, then

$$\frac{R_0^{HM}}{R_0^{CM}} = \frac{\alpha/\mathbb{E}[\bar{D}-1]+\gamma}{\gamma} > 1,$$

which proves that $R_0^{HM} \geq R_0^{CM}$.

3. Consider an SIR epidemic Markov model for a large homogeneous closed population in which people form couples, are serial monogamous (i.e. a person is in at most one couple at a time) and can only transmit within a couple.

More specific: Consider a population of size N. Initially there are $N_1(0)$ single people and $N_2(0)$ couples of people in a relationship (So, $N_1(0)+2N_2(0) = N$). Existing couples break up at rate ν and two single people form a couple at per pair of singles rate κ/N. Initially there is 1 couple consisting of a susceptible and an infectious person, while all other people (single or part of a couple) are susceptible. Within couples contacts occur according to independent Poisson processes with intensity β. If the contact is between a susceptible and infectious person, the susceptible becomes infectious immediately. Infectious people recover (and become immune forever) independently at per person rate γ. The process is a Markov process.

a) What is the probability that the susceptible person in an SI couple becomes infected before the pair breaks up? (10pt)

b) Use a deterministic approximation to find $\hat{x}$, which is the asymptotic (as $N \to \infty$) fraction of people that is single. (10pt)

Hint: Let $N_1 = N_1(t)$ be the number of single people and $x = x(t) = N_1/N$. Find for which x the rate of pair formation is equal to the rate of breaking of pairs.

c) Consider a person that is just infected. Approximate the process (no justification needed) in such a way that this person when single, becomes part of a couple at rate $\kappa\hat{x}$. Provide the distribution of the number of couples this person is part of during their infectious period. (10pt)

d) Using the same approximation as for part c, provide an expression for the basic reproduction number R_0 if $N_1(0)/N = \hat{x}$, and justify your answer. (5pt)

Solution

a) A susceptible in an SI couple becomes infected before the couple breaks up, if the first contact between the couple takes place before the infectious person in the couple recovers or when this contact does not take place at all, because the couple already broke up. The combined rate of recovery and breaking up is $\gamma + \nu$. So, the probability of infection before breaking up is

$$\int_0^\infty (\gamma+\nu)e^{-(\gamma+\nu)t}(1-e^{-\lambda t})dt = \frac{\lambda}{\gamma+\nu+\lambda}.$$

b) Note that $N_1(t)+2N_2(t) = N$ and the rate at which $N_1(t)$ increases by 2 is $\nu N_2(t) = \nu(N-N_1(t))/2$ and the rate which $N_1(t)$ decreases by 2 is $\frac{\kappa}{N}N_1(t)(N_1(t)-1)/2$. So, the deterministic approximation we obtain

$$\frac{d}{dt}N_1(t) = 2\times \nu(N-N_1(t))/2 - 2\times\frac{\kappa}{N}N_1(t)(N_1(t)-1)/2$$

dividing by N we obtain (noting that $1/N \to 0$)

$$\frac{dx}{dt} = \nu(1-x) - \kappa x^2.$$

Setting the right hand side equal to 0, we obtain that $\hat{x}$ should satisfy.

$$\kappa\hat{x}^2 + \nu\hat{x} - \nu = 0,$$

So, $\hat{x} = \frac{-\nu\pm\sqrt{\nu^2+4\nu\kappa}}{2\kappa}$ and the only positive solution is

$$\hat{x} = \sqrt{(\nu/(2\kappa))^2 + (\nu/\kappa)} - \nu/(2\kappa).$$

c) Let X be distributed as the number of couples this person is part of during their infectious period. The probability that an infectious person, which is now part of a couple recovers before becoming part of the next couple is

$$p := 1 - \frac{\nu}{\nu+\gamma}\times\frac{\kappa\hat{x}}{\kappa\hat{x}+\gamma} = \frac{(\nu+\gamma)(\kappa\hat{x}+\gamma)-\nu\kappa\hat{x}}{(\nu+\gamma)(\kappa\hat{x}+\gamma)} = \frac{\gamma(\kappa\hat{x}+\gamma+\nu)}{(\nu+\gamma)(\kappa\hat{x}+\gamma)}.$$

The process is a Markov process, so $\mathbb{P}(X > k|X \geq k) = (1-p)$, at the moment of infection a person is part of a couple, so $\mathbb{P}(X \geq 1) = 1$. This implies that X is geometrically distributed with $\mathbb{P}(X = k) = p(1-p)^{k-1}$ for $k \in \mathbb{N}_{\geq 0}$.

d) A typical infected person (say v) during the early stages of the epidemic is (in expectation) part of $\mu_X = \mathbb{E}[X]$ couples during their infectious period. The partner of x in the first couple is necessarily infectious (because it is the infector of x), but the other partners are in the large population limit with high probability susceptible at time of couple formation. Each of those other partners gets infected while being a couple with x with probability $\frac{\lambda}{\gamma+\nu+\lambda}$. So

$$R_0 = \frac{\lambda}{\gamma+\nu+\lambda}(\mu_X - 1) = \frac{\lambda}{\gamma+\nu+\lambda}(p^{-1}-1) = \frac{\lambda}{\gamma+\nu+\lambda}\frac{\nu\kappa\hat{x}}{\gamma(\kappa\hat{x}+\gamma+\nu)}.$$